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Generalization of the Larmor Radiation Formula

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The Larmor radiation formula relates the momentum radiated toward infinity along a light cone from a single point charge to the velocity and acceleration of the particle. The formula applies only to a light cone whose apex is on the world line of the particle. This paper generalizes the Larmor formula to an arbitrary light cone. An example involving circular motion is worked.

1. INTRODUCTION

The Larmor radiation formula says that if a particle with electric charge e moves along a world line $\mathbf{Z}^a(\cdot)$ parametrized by its arc length, then

$$\tfrac{2}{3}e^2(\ddot{\mathbf{Z}}^b\ddot{\mathbf{Z}}_b)\dot{\mathbf{Z}}^a = \lim_{R\to\infty}\lim_{\varepsilon\to 0}\frac{c}{\varepsilon}\int_{S_R}\mathbf{T}^{ab}\,d\sigma_a \tag{1.1}$$

where $\mathbf{T}^{ab}$ is the energy-momentum tensor. Assuming that $\mathbf{Z}^a(0)=0$, the surface S_R is a cylinder parallel to the timelike vector $\mathbf{l}^a$ defined by $\mathbf{l}^a \equiv \dot{\mathbf{Z}}^a(0)$. I will denote the forward light cone with apex at $\mathbf{a}$ by $L^+(\mathbf{a})$, and the backward light cone by $L^-(\mathbf{a})$. The edges of the cylinder S_R are on the light cones $L^+(0)$ and $L^+(\varepsilon\mathbf{l}^a)$. The edge of S_R that is on the light cone $L^+(0)$ is the intersection of the hyperplane $\mathbf{X}^a\mathbf{l}_a = R$ with the cone $L^+(0)$. With these assumptions the field on the $L^+(0)$ is completely determined by $\dot{\mathbf{Z}}^a(0)$ and by $\ddot{\mathbf{Z}}^a(0)$.

This paper generalizes the Larmor result by removing the condition $\mathbf{Z}^a(0)=0$. Without this condition it is no longer true that the field on $L^+(0)$ is determined by only $\dot{\mathbf{Z}}^a(0)$ and $\ddot{\mathbf{Z}}^a(0)$. In this case, the field on $L^+(0)$ is determined by the portion of the world line that is spacelike-separated from the origin (Willis, 1989). I will assume that the world line intersects both the

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forward and backward light cones with apex at 0. Thus, if $\tau^{\pm}$ are defined by $\mathbf{Z}^a(\tau^{\pm}) \in L^{\pm}(0)$ with $\tau^- < \tau^+$, then the field on $L^+(0)$ is determined by the finite portion of the world line $\mathbf{Z}^a(\tau)$ with $\tau \in [\tau^-, \tau^+]$.

In this case, it is reasonable to assume that the radiated momentum can be expressed as an integral over the finite portion of the world line that is spacelike-separated from the origin. This paper gives such a result.

2. PARAMETRIZATION OF S_R

In order to express the radiated momentum as an integral over the world line, it is necessary to parametrize the surface S_R using the retarded proper time of the world line. If $\mathbf{X}^a \in S_R$, then the retarded proper time τ is defined implicitly by

$$[\mathbf{X}^a - \mathbf{Z}^a(\tau)][\mathbf{X}_a - \mathbf{Z}_a(\tau)] = 0 \tag{2.1}$$

The other two parameters are an angle φ and a length s. The parameter s will generate the lateral portion of S_R. Since the cylinder S_R axis is the timelike vector $\mathbf{l}^a$, the parametrization has the form

$$\mathbf{X}^a(\tau, \varphi, s) = \mathbf{U}^a(\tau, \varphi) + s\mathbf{l}^a \tag{2.2}$$

The function $\mathbf{U}^a$ parametrizes the two-dimensional surface formed by the intersection of the cone $\mathbf{X}^a\mathbf{X}_a = 0$ and by the hyperplane $\mathbf{X}^a\mathbf{l}_a = R$. Thus the vector $\mathbf{U}^a$ must satisfy

$$\mathbf{U}^a(\tau, \varphi)\mathbf{U}_a(\tau, \varphi) = 0 \tag{2.3a}$$

$$[\mathbf{U}^a(\tau, \varphi) - \mathbf{Z}^a(\tau)][\mathbf{U}_a(\tau, \varphi) - \mathbf{Z}_a(\tau)] = 0 \tag{2.3b}$$

$$\mathbf{U}^a(\tau, \varphi)\mathbf{l}_a = R \tag{2.3c}$$

Equation (2.3b) simplifies to

$$-2\mathbf{U}^a\mathbf{Z}_a + \mathbf{Z}^a\mathbf{Z}_a = 0 \tag{2.4}$$

Equations (2.3c) and (2.4) imply that the vector $\mathbf{U}^a$ can be expressed as

$$\mathbf{U}^a = \mathbf{W}^a + \mu_1\mathbf{R}^a + \mu_2\mathbf{S}^a \tag{2.5}$$

where $\mathbf{W}^a \in \operatorname{span}\{\mathbf{Z}^a(\tau), \mathbf{l}^a\}$, the vectors $\mathbf{R}^a$ and $\mathbf{S}^a$ are perpendicular to $\operatorname{span}\{\mathbf{Z}^a(\tau), \mathbf{l}^a\}$, and $\mathbf{R}^a \perp \mathbf{S}^a$. Since both $\mathbf{R}^a$ and $\mathbf{S}^a$ are perpendicular to the timelike vector $\mathbf{l}^a$, it follows that $\mathbf{R}^a$ and $\mathbf{S}^a$ are spacelike. I will assume that they are normalized so that

$$\mathbf{R}^a\mathbf{R}_a = -1 \tag{2.6a}$$

$$\mathbf{S}^a\mathbf{S}_a = -1 \tag{2.6b}$$

$$\mathbf{S}^a\mathbf{R}_a = 0 \tag{2.6c}$$

Since $\mathbf{S}^a$ is spacelike, it is consistent to require it to be perpendicular to the timelike vector $\dot{\mathbf{Z}}^a$. Thus, $\mathbf{S}^a$ is perpendicular to the vectors $\mathbf{Z}^a$, $\dot{\mathbf{Z}}^a$, and $\mathbf{l}^a$. So, $\mathbf{S}^a$ is given by

$$\mathbf{S}^a = \frac{1}{d_1} \varepsilon^{abcd} \mathbf{Z}_b \dot{\mathbf{Z}}_c \mathbf{l}_d \tag{2.7}$$

The constant d_1 is determined by the normalization condition $\mathbf{S}^a\mathbf{S}_a = -1$. The vector $\mathbf{R}^a$ is perpendicular to $\mathbf{Z}^a$, $\mathbf{l}^a$, and $\mathbf{S}^a$. Therefore, $\mathbf{R}^a$ is given by

$$\mathbf{R}^a = \frac{1}{d_2} \varepsilon^{abcd} \mathbf{Z}_b \mathbf{S}_c \mathbf{l}_d \tag{2.8}$$

where the constant d_2 is determined by the normalization condition $\mathbf{R}^a\mathbf{R}_a = -1$.

The condition $\mathbf{U}^a\mathbf{U}_a = 0$ restricts the scalars μ_1 and μ_2 [see Eq. (2.5)]. This condition is

$$\mu_1^2 + \mu_2^2 = \mathbf{W}^a\mathbf{W}_a \tag{2.9}$$

Solving the linear equations for $\mathbf{W}$ gives the result

$$\mathbf{W}^a\mathbf{W}_a = \frac{\mathbf{Z}^a\mathbf{Z}_a}{(\mathbf{Z}^a\mathbf{l}_a)^2 - \mathbf{Z}^a\mathbf{Z}_a} \tag{2.10}$$

I now define the angle φ by

$$\mu_1 = (\mathbf{W}^a\mathbf{W}_a)^{1/2} \cos(\varphi) \tag{2.11a}$$

$$\mu_2 = (\mathbf{W}^a\mathbf{W}_a)^{1/2} \sin(\varphi) \tag{2.11b}$$

where $\varphi \in [0, 2\pi)$. Thus, the form of the parametrization is

$$\begin{aligned} \mathbf{X}^a(\tau, \varphi, s) = {} & \mathbf{W}^a(\tau) + (\mathbf{W}^b\mathbf{W}_b)^{1/2} \\ & \times [\cos(\varphi)\mathbf{R}^a(\tau) + \sin(\varphi)\mathbf{S}^a(\tau)] + s\mathbf{l}^a \end{aligned} \tag{2.12}$$

In the next section the Jacobian of this parametrization will be computed.

3. THE JACOBIAN

In this section the Jacobian $\mathbf{j}^a$ given by

$$\mathbf{j}^a = \pm \varepsilon^{abcd} \mathbf{Z}_{b,\tau} \mathbf{X}_{c,\varphi} \mathbf{X}_{d,s} \tag{3.1}$$

will be computed. The sign of $\mathbf{j}^a$ is chosen so that it is an outward normal.

This task of computing $\mathbf{j}^a$ is simplified by using the fact that the direction of $\mathbf{j}^a$ is independent of the particular parametrization. Using a simple parametrization of S_R, one can show that there is a scalar Φ so that

$$\mathbf{j}^a = \Phi[\mathbf{X}^a - (\mathbf{X}^b \mathbf{l}_b)\mathbf{l}^a] \tag{3.2}$$

Since the limit as $\varepsilon \to 0$ is to be evaluated [see Eq. (1.1)], it follows that Φ needs only to be computed to nonvanishing order in ε. Simplifying $\mathbf{j}^a \mathbf{X}_a$ shows that Φ is given by

$$\Phi = \pm \frac{1}{R^2} \mathbf{j}^a \mathbf{X}_a \tag{3.3a}$$

$$= \pm \frac{1}{R^2} \varepsilon^{abcd} \mathbf{X}_a \mathbf{X}_{b,\tau} \mathbf{X}_{c,\varphi} \mathbf{X}_{d,s} \tag{3.3b}$$

The partial derivatives of $\mathbf{X}^a$ with respect to τ and φ are determined by implicit differentiation of the equations

$$\mathbf{X}^a \mathbf{X}_a = 0 \tag{3.4a}$$

$$\mathbf{X}^a \mathbf{l}_a = R \tag{3.4b}$$

$$\mathbf{X}^a \mathbf{Z}_a = \tfrac{1}{2} \mathbf{Z}^a \mathbf{Z}_a \tag{3.4c}$$

$$\mathbf{X}^a \dot{\mathbf{Z}}_a = \mathbf{W}^a \dot{\mathbf{Z}}_a + (\mathbf{W}^a \mathbf{W}_a)^{1/2} \mathbf{R}^a \dot{\mathbf{Z}}_a \cos(\varphi) \tag{3.4d}$$

Defining the scalars $a(\tau)$ and $b(\tau)$ by

$$a(\tau) = \mathbf{W}^a \dot{\mathbf{Z}}_a \tag{3.5a}$$

$$b(\tau) = (\mathbf{W}^b \mathbf{W}_b)^{1/2} \mathbf{R}^a \dot{\mathbf{Z}}_a \cos(\varphi) \tag{3.5b}$$

the required derivatives of $\mathbf{X}$ can be expressed as combination of the vectors $\mathbf{A}^a$, $\mathbf{B}^a$, and $\mathbf{C}^a$ defined by

$$\mathbf{A}^a = \varepsilon^{abcd} \mathbf{X}_b \mathbf{Z}_c \dot{\mathbf{Z}}_d \tag{3.6a}$$

$$\mathbf{B}^a = \varepsilon^{abcd} \mathbf{X}_b \dot{\mathbf{Z}}_c \mathbf{l}_d \tag{3.6b}$$

$$\mathbf{C}^a = \varepsilon^{abcd} \mathbf{X}_b \mathbf{Z}_c \mathbf{l}_d \tag{3.6c}$$

The derivatives also involve scalars ρ, q, and Δ defined by

$$\Delta = \varepsilon^{abcd} \mathbf{l}_a \mathbf{X}_b \mathbf{Z}_c \dot{\mathbf{Z}}_d \tag{3.7a}$$

$$\rho = (\mathbf{X}^a - \mathbf{Z}^a) \dot{\mathbf{Z}}_a \tag{3.7b}$$

$$q = (\mathbf{X}^a - \mathbf{Z}^a) \ddot{\mathbf{Z}}_a \tag{3.7c}$$

I find that

$$\mathbf{X}_{a,\tau}=\frac{\rho}{\Delta}\mathbf{B}_a+\frac{1}{\Delta}[q-\dot{a}-\dot{b}\cos(\varphi)]\mathbf{C}_a \tag{3.8a}$$

$$\mathbf{X}_{a,\varphi}=\frac{b}{\Delta}\sin(\varphi) \tag{3.8b}$$

$$\mathbf{X}_{a,s}=\mathbf{l}_a \tag{3.8c}$$

Simplifying Φ, I find that

$$\Phi=\frac{b\rho}{\Delta}\sin(\varphi) \tag{3.9}$$

This can be further simplified by using the formula for the contraction of two Levi-Civita (Wald, 1984) symbols to simplify Δ, $b(\tau)$, and d_2. I find that

$$\Delta=d_1(\mathbf{W}^a\mathbf{W}_a)^{1/2}\sin(\varphi) \tag{3.10a}$$

$$b(\tau)=\frac{d_1}{d_2}(\mathbf{W}^a\mathbf{W}_a)^{1/2} \tag{3.10b}$$

$$d_2=[(\mathbf{Z}^a\mathbf{l}_a)^2-\mathbf{Z}^a\mathbf{Z}_a\mathbf{l}^b\mathbf{l}_b]^{1/2} \tag{3.10c}$$

Therefore, Φ is given by

$$\Phi=\frac{\rho}{[(\mathbf{Z}^a\mathbf{l}_a)^2-\mathbf{Z}^a\mathbf{Z}_b\mathbf{l}^b\mathbf{l}_b]^{1/2}} \tag{3.11}$$

In the next section an example involving circular motion is worked.

4. CIRCULAR MOTION

In this section an example involving circular motion is worked. I will assume that the world line is parametrized by

$$\mathbf{Z}^a(\tau)=\langle(1+\beta^2)^{1/2}\tau,\ a\cos(\omega\tau),\ a\sin(\omega\tau),\ 0\rangle \tag{4.1}$$

where $\beta \in (0, 1)$, $a > 0$, and $a\omega = \beta$. I also assume that $\mathbf{l}^a = \langle 1, 0, 0, 0 \rangle$. For this motion, one can show that

$$\tau^{\pm} = \frac{a}{(1+\beta^2)^{1/2}} \tag{4.2a}$$

$$\rho = R\left\{(1+\beta^2)^{1/2} + \beta\left[1 - (1+\beta^2)\left(\frac{\tau}{a}\right)^2\right]^{1/2} \cos(\varphi)\right\} \tag{4.2b}$$

$$q = \frac{R}{a}\beta^2(1+\beta^2)^{1/2}\tau \tag{4.2c}$$

$$\Phi = \frac{\rho}{a} \tag{4.2d}$$

$$\mathbf{R}^a(\tau) = \langle 0, \sin(\omega\tau), -\cos(\omega\tau), 0 \rangle \tag{4.2e}$$

$$\mathbf{S}^a(\tau) = \langle 0, 0, 0, 1 \rangle \tag{4.2f}$$

If $\mathbf{T}^{ab}$ is the radiation part of the energy-momentum tensor, then one can show that $\mathbf{T}^{ab}\mathbf{j}_a\mathbf{l}_b$ is given by

$$\frac{e^2}{4\pi}\frac{\beta^4}{a^2}\left(\frac{s^2}{[(1+\beta^2)^{1/2} + \beta(1-s^2)^{1/2}\cos(\varphi)]^5} - \frac{1}{[(1+\beta^2)^{1/2} + \beta(1-s^2)^{1/2}\cos(\varphi)]^3}\right) \tag{4.3}$$

where $as = \tau(1+\beta^2)^{1/2}$. Introducing the Jacobian of this change of variable and integrating over $\varphi \in [0, 2\pi)$ and $s \in [-1, 1]$ gives the result for the radiated power P,

$$P = \frac{2}{3}\frac{\beta^4}{a^2} \tag{4.4}$$

5. DISCUSSION

Most derivations of Dirac's equation of motion with radiation damping rely on the Larmor radiation formula. The generalization of the Larmor formula given in this paper might be useful in understanding the equations of motion for point charges.

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